7.5 (continued)

last time : $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\underbrace{\mathcal{L}\{f(t)\}}$

transform after shifting $f(t-c)$
back to origin LEFT by c
t turns into $t+c$

for example, $\mathcal{L}\{u_{10}(t)e^{-2t}\}$

$$= e^{-10s}\mathcal{L}\{e^{-2(t+10)}\}$$

$$= e^{-10s}e^{-20}\mathcal{L}\{e^{-2t}\}$$

$$= e^{-10s}e^{-20}\frac{1}{s+2}$$

t to s: shift LEFT ($t \to t+c$), transform, $u_c \to e^{-cs}$

back to t is above in reverse/opposite

s to t: $e^{-cs} \to u_c$, inverse transform, shift RIGHT ($t \to t-c$)

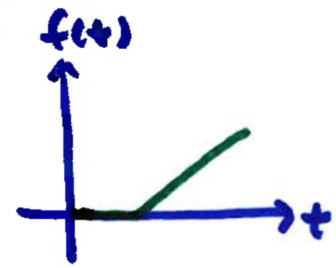
for example, $\mathcal{L}^{-1}\left\{e^{-\pi s}\frac{2}{s^3}\right\}$ $\qquad$ $\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = t^2$

$= u_\pi(t)\cdot(t-\pi)^2$ $\qquad$ Shifted RIGHT by π

full example of solving diff. eq.

$y'' + y = f(t)$ $\qquad$ $y(0) = y'(0) = 0$

$$f(t) = \begin{cases} 0 & 0 \le t < 3 \\ t-3 & t \ge 3 \end{cases}$$



transform: $\qquad$ $f(t) = u_3(t)\cdot(t-3)$

$s^2 Y - s\cancel{y(0)} - \cancel{y'(0)} + Y = F(s)$ *

$(s^2+1)Y = e^{-3s}\mathcal{L}\{(\underbrace{t+3}_{}) - 3\}$

shift LEFT by 3 : $t \to t+3$

$(s^2+1)Y = e^{-3s}\frac{1}{s^2}$

$$Y = e^{-3s}\frac{1}{s^2(s^2+1)} \longrightarrow \frac{As+B}{s^2} + \frac{Cs+D}{s^2+1} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

preliminary inverse transform

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+1}\right\} = t - \sin(t)$$

back to t: $e^{-3s} \to u_3(t)$

inv. transform, shift RIGHT by 3 : $t \to t-3$

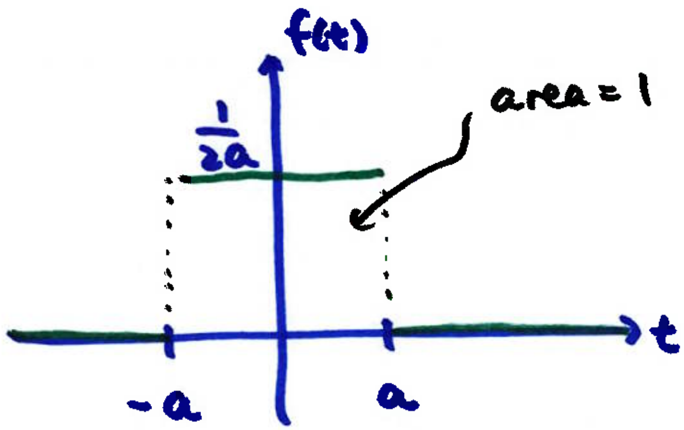
$$y(t) = u_3(t)\left[(t-3) - \sin(t-3)\right]$$

$$= \begin{cases} 0 & 0 \le t < 3 \\ t-3-\sin(t-3) & t \ge 3 \end{cases}$$

7.6 Impulse Function

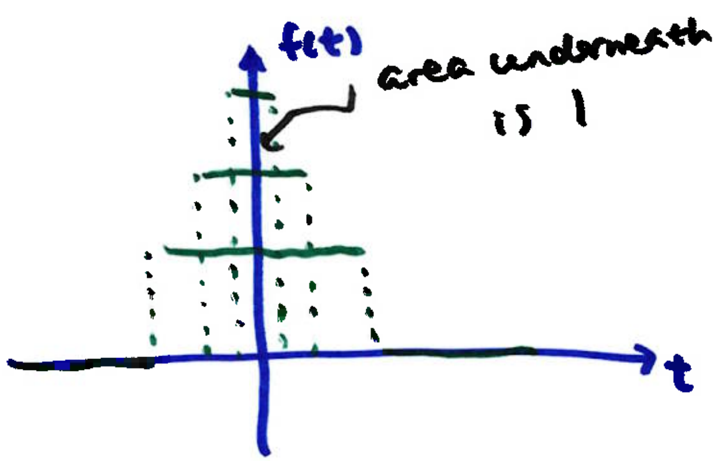
used to model a short-acting input (e.g. kicking a ball)

we can construct an impulse function from unit step functions

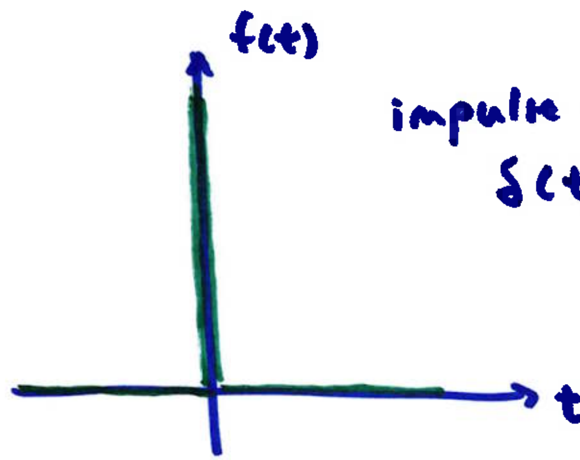


$$f(t) = \frac{1}{2a}\left[u_{-a}(t) - u_{a}(t)\right]$$

now shrink a: $\lim_{a \to 0} f(t)$



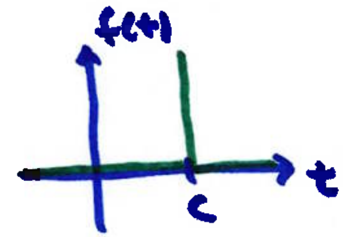
$\rightarrow$



impulse function $\delta(t)$

$$\delta(t) = \begin{cases} +\infty & \text{if } t = 0 \\ 0 & \text{else} \end{cases} \qquad \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

$$\delta(t-c) = \begin{cases} -\infty & \text{if } t = c \\ 0 & \text{else} \end{cases}$$



let's find $\mathcal{L}\{\delta(t-c)\}$

$$= \mathcal{L}\left\{ \lim_{a \to 0} \left[u_{c-a}(t) \cdot \frac{1}{2a} - u_{c+a}(t) \cdot \frac{1}{2a} \right] \right\}$$

$$= \lim_{a \to 0} \frac{1}{2a} \left[\mathcal{L}\{u_{c-a}(t)\} - \mathcal{L}\{u_{c+a}(t)\} \right]$$

$$= \lim_{a \to 0} \frac{1}{2a} \left[\frac{e^{-(c-a)s}}{s} - \frac{e^{-(c+a)s}}{s} \right]$$

$$= \frac{1}{s} \lim_{a \to 0} \frac{e^{-(c-a)s} - e^{-(c+a)s}}{2a} \qquad \text{l'Hospital's Rule}$$

$$= \frac{1}{s} \lim_{a \to 0} \frac{e^{-cs} e^{+as} - e^{-cs} e^{-as}}{2a}$$

$$= \frac{e^{-cs}}{s} \lim_{a \to 0} \frac{e^{as} - e^{-as}}{2a}$$

$$= \frac{e^{-cs}}{s} \lim_{a \to 0} \frac{s e^{as} + s e^{-as}}{2}$$

$$= \frac{e^{-cs}}{s} \cdot s = e^{-cs}$$

$$\boxed{\mathcal{L}\{\delta(t-c)\} = e^{-cs}}$$

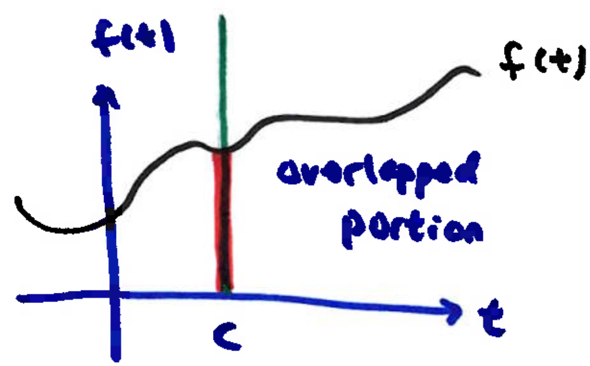
looks like unit step

$$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}$$

$$\delta(t-c) = \begin{cases} +\infty & t = c \\ 0 & \text{else} \end{cases}$$

$$\delta(t-c) f(t) = \begin{cases} f(c) & t = c \\ 0 & \text{else} \end{cases}$$

"sampled impulse"



$$\mathcal{L}\{\delta(t-c)f(t)\} = f(c)e^{-cs}$$

revisit earlier example:

$$y'' + y = f(t) \qquad y(0) = y'(0) = 0$$

$$f(t) = \delta(t-3)$$

$$y'' + y = \delta(t-3)$$

$$s^2Y + Y = e^{-3s}$$

$$Y = e^{-3s}\frac{1}{s^2+1}$$

$e^{-3s} \to u_3$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin(t)$

shift RIGHT by 3 : $t \to t-3$

$$y(t) = u_3(t) \cdot \sin(t-3)$$

does NOT come back to t as impulse

because effect due to impulse doesn't go away
(kicking a ball the ball flies forever until another input)